

# Intermittent Redesign of Analog Controllers via the Youla Parameter

Leonid Mirkin

**Abstract**—The paper studies digital redesign of linear time-invariant analog controllers under intermittent sampling. The sampling pattern is only assumed to be uniformly bounded, but otherwise irregular and unknown a priori. The contribution of the paper is twofold. First, it proposes a constructive algorithm to redesign any analog stabilizing controller so that the closed-loop stability is preserved. Second, it is shown that when applied to (sub) optimal  $H^2$  and  $H^\infty$  controllers, the algorithm produces (sub) optimal sampled-data solutions under any a priori unknown sampling pattern. The proposed solutions are analytic, computationally simple, implementable, and transparent. Transparency pays off in showing the optimality, under a fixed sampling density, of uniform sampling for both performance measures studied.

**Index Terms**—Sampled-data systems, intermittent sampling, Youla-Kučera parametrization,  $H^2$  and  $H^\infty$  optimization

## I. INTRODUCTION

The term “digital redesign” refers to problems of approximating analog controllers by sampled-data ones, i.e. controllers that can be realized as the cascade of a sampler (A/D converter), a pure discrete element, and a hold (D/A converter) as shown in Fig. 1. This approach has been widely employed in designing digital controllers for analog plants, not least because it facilitates the direct use of analog insights in the design. The reader is referred to [1, Ch. 8] and [2, Ch. 3] for expositions of ideas in the field and further references.

A common digital redesign setup is to assume a regular (say, constant) sampling rate, fixed A/D and D/A parts (say, the ideal sampler and the zero-order hold, respectively, as in Fig. 1), and choose a discrete-time part that mimics the structure of the analog prototype. But these choices are, to some extent, a legacy of technological and methodological limitations of early computer-controlled systems. Nowadays, with the advent of affordable DSP technology and a trend to distribute information processing, the accents are changing.

First, the use of traditional A/D and D/A converters might no longer be preordained. There may be enough local computational power to pre-process measurements and post-process control commands. Model-based modifications of control signal during the intersample, dubbed the generalized hold, were exploited in [3] (in fact, an application of a generalized hold mechanism to the digital redesign problem was already proposed in [4]), with the philosophy to circumvent limitations of linear control. This philosophy was then criticized in [5]. Optimal design of generalized sampler and hold, which are not prone to the problems presented in [5], was pioneered in [6], see also [7]. Lately, there is a renewed interest in this subject, see e.g. [8, 9] and the references therein.

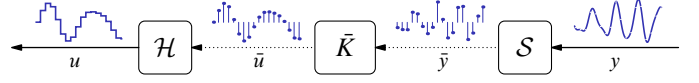


Fig. 1. Generic sampled-data controller as the cascade of an A/D converter (sampler)  $S$ , a pure discrete-time part  $\bar{K}$ , and a D/A converter (hold)  $\mathcal{H}$

Second, there have been rapidly growing activities in systems with intermittent sampling. This is motivated by networked control systems [9] and potential advantages in employing event-based feedback [8, 10]. Although the results might not study digital redesign explicitly (an exception is [11]), many of them effectively deal with these problems. Of a special interest for us are approaches that make use of the simulated analog closed-loop system to generate control signals during intersample intervals of irregular lengths and, if not the whole state is measured, adjust an analog state estimator upon arrival of new samples. This direction is exposed in [9]. See also [12] for apparently the first appearance of such an idea in the control literature and [13] and the references therein for its use in human control, although these two references offer neither proofs of stability nor performance analyses. It is worth emphasizing that many methods, which use intermittent sampling, augment the original analog controller, so that its discretized version may be more complex. This departure from the conventional *modus operandi* reflects the changing accents mentioned above: more emphasis is placed on the information exchange between system components and the form of A/D and D/A converters is less restrictive.

Tackling systems with intermittent sampling events might be a challenge, owing to their time-varying nature and switches between closed- and open-loop regimes. This is true in handling the closed-loop stability and even more so in analyzing performance. Consequently, results are frequently either conservative or apply only to simple dynamics. The full access to the plant state is a recurrent assumption. There appear to be no non-conservative and transparent methods of optimal control design for general linear problems with general sampling patterns. Besides, although the use of unorthodox hold and sampling elements has proved useful, their structures are often justified only empirically. The apparent qualitative difference from systems with periodic sampling brought about different analysis tools, like continuous-time Lyapunov methods.

One of the goals of this paper is to demonstrate that concepts and tools developed for sampled-data problems with periodic sampling might still be powerful in addressing stability and performance problems under intermittent sampling. It is shown that the ideas of [14], which exploit properties of conventional sampled-data systems in the *lifted domain*, extend to systems

with intermittent sampling. Specifically, [14] shows that the set of all causal finite-dimensional sampled-data systems corresponds to the set of strictly causal systems in the lifted domain. This result facilitates extracting sampled-data controllers from various analog controller parametrizations. By extending the result to the intermittent sampling setup, the following set of redesign problems is addressed:

- 1) An approach to digitally redesign given analog stabilizing controllers is put forward. By embedding such controllers into the analog Youla parametrization setup, all stabilizing sampled-data controllers are characterized. This yields a systematic algorithm to construct a stabilizing controller under any, even unknown a priori, sampling pattern.
- 2) Intermittent redesign methods for analog  $H^2$  and  $H^\infty$  (sub) optimal controllers are proposed. They result in non-conservative optimal designs under no limitation on the sampling pattern. Performance levels attainable by the resulting sampled-data controllers are transparent functions of sampling times. As a result, it is proved that the uniform sampling is both  $H^2$  and  $H^\infty$  optimal among all sampling patterns of a given density.

Remarkably, the offline computational complexity of the algorithms above is independent of the sampling pattern. Also, the resulting sampler and hold are justified performance wise. To the best of my knowledge, these are the first non-conservative and computationally tractable results for general linear problems with unrestricted sampling patterns.

The paper is organized as follows. After presenting some preliminary results about the Youla parametrization and lifting in Section II, the class of sampled-data controllers is characterized in the lifted domain in Section III. This result is then used to address the stabilization problem in Section IV, where a parametrization of all sampled-data stabilizing controllers for an arbitrary sampling pattern is presented (Theorem 4.2) and some of their properties are discussed. The next section is devoted to the performance-based discretizations, in the  $H^2$  (§V-A) and  $H^\infty$  (§V-B) senses. Section VI shows how the proposed approach can be applied to the  $H^\infty$  loop shaping method of [15] and illustrates this procedure by a numerical example. Concluding remarks are provided in Section VII and the Appendix contains some more technical proofs.

*Notation:* The sets of non-negative integers and reals are denoted as  $\mathbb{Z}^+$  and  $\mathbb{R}^+$ , respectively. The transpose of a matrix  $M$  is denoted as  $M'$  and, for square matrices,  $\text{tr}(M)$  and  $\rho(M)$  stand for the trace and the spectral radius of  $M$ .  $\mathcal{F}_l(\Phi, \Omega)$  and  $\mathcal{F}_u(\Phi, \Omega)$  read as the lower and upper linear-fractional transformations of  $\Omega$  by  $\Phi$ , respectively, see [16, Ch. 10].

## II. PRELIMINARIES

This section revises the Youla parametrization and the lifting technique, which are required for technical developments in the paper. Although both subjects are well-studied in the literature, both require some less documented twists.

### A. Youla parametrization with prespecified central controller

Parametrizations of all stabilizing controllers for a given LTI plant, known as the Youla, or Youla-Kučera, parametrizations,

is a classical result, well documented in the literature, see [16, Ch. 12] and the references therein. The idea also extends to time-varying systems [17, Sec. 9.A]. These parametrizations are conventionally expressed in terms of a linear-fractional transformation of a free stable parameter (dubbed the “ $Q$ -parameter”) by some given “generator,” which is a function of a coprime factorization of the plant. When state-space realizations are involved, the central controller, the one corresponding to  $Q = 0$ , has commonly the observer-based structure.

It is less common to construct a parametrization centered on some given, “nominal,” stabilizing controller, which is not necessarily observer based. This possibility was explored in [18, §III-B] in the case when this nominal stabilizing controller is stable itself. An insight into how to expand a given controller was provided in [19, pp. 546–548], but constructive procedures and completeness were only discussed for stable plants with the zero nominal controller and for observer-based nominal controllers. I am not aware of other discussions of this subject in the literature. Still, this kind of parametrization is required for developments in the next section. Thus, although the result might not be entirely new, it is proved below.

*Lemma 2.1:* Let  $P$  be an LTI plant having a strictly proper transfer function. Assume that it is internally stabilized by an LTI finite-dimensional controller  $K_0$ . Then all linear internally stabilizing controllers can be characterized as  $K = \mathcal{F}_l(J_0, Q)$  for any stable and causal  $Q$  and

$$J_0 := \begin{bmatrix} K_0 & \tilde{M}_0^{-1} \\ M_0^{-1} & -M_0^{-1}P(I - K_0P)^{-1}\tilde{M}_0^{-1} \end{bmatrix}, \quad (1)$$

where  $M_0$ ,  $N_0$ ,  $\tilde{M}_0$ , and  $\tilde{N}_0$  are coprime factors of  $K_0$  over  $RH^\infty$ , such that  $K_0 = \tilde{M}_0^{-1}\tilde{N}_0 = N_0M_0^{-1}$ .

*Proof:* Because  $K_0$  is stabilizing, there must exist coprime factorizations of the plant,  $P = \tilde{M}_P^{-1}\tilde{N}_P = N_PM_P^{-1}$ , such that

$$\begin{bmatrix} \tilde{M}_0 & -\tilde{N}_0 \\ -\tilde{M}_P & \tilde{N}_P \end{bmatrix} \begin{bmatrix} M_P & N_0 \\ N_P & M_0 \end{bmatrix} = I.$$

Indeed, by [16, Lem. 5.10] the stability of the closed-loop system implies that for any coprime factorizations of  $P$ , the systems  $D := \tilde{M}_0M_P - \tilde{N}_0N_P$  and  $\tilde{D} := \tilde{M}_PM_0 - \tilde{N}_PN_0$  are bi-stable, i.e., such that  $D, D^{-1}, \tilde{D}, \tilde{D}^{-1} \in H^\infty$ . Thus,  $M_PD^{-1}$ ,  $N_PD^{-1}$ ,  $\tilde{D}^{-1}\tilde{M}_P$ , and  $\tilde{D}^{-1}\tilde{N}_P$  are also coprime factors of  $P$  and they do verify the equality above. It then follows from [17, Sec. 9.A] that all internally stabilizing controllers can be parametrized as  $(N_0 + M_PQ)(M_0 + N_PQ)^{-1}$ . The equivalence between this form and (1) follows by [16, Lem. 10.1] and the fact that  $P(I - K_0P)^{-1} = N_P\tilde{M}_0$ . Finally, as  $P(\infty) = 0$ , the (2, 2) sub-block of  $J_0(s)$  is strictly proper and the LFT in (1) is well posed for every causal  $Q$  by [20, Thm. 4.1].  $\square$

By [16, Lem. 10.4(c)] the transformation  $Q \mapsto K$  defined by (1) is invertible, with  $Q = \mathcal{F}_u(J_0^{-1}, K)$ , where

$$J_0^{-1} = \begin{bmatrix} P & M_0 - PN_0 \\ \tilde{M}_0 - \tilde{N}_0P & -\tilde{N}_0(M_0 - PN_0) \end{bmatrix}, \quad (2)$$

and is well posed for any causal  $K$ , again by [20, Thm. 4.1].

*Remark 2.1 (connection with [18]):* The parametrization of Lemma 2.1 can be rewritten as

$$K = \mathcal{F}_l \left( \begin{bmatrix} K_0 & I \\ I & -P(I - K_0P)^{-1} \end{bmatrix}, \hat{Q} \right),$$

where  $\hat{Q} := \tilde{M}_0^{-1} Q M_0^{-1}$ . If  $K_0$  is stable, both  $\tilde{M}_0$  and  $M_0$  are bi-stable and can thus be absorbed into the  $Q$ -parameter. The parametrization then reduces to the case discussed in [18]. Yet unstable poles of  $\tilde{M}_0^{-1}$  and  $M_0^{-1}$ , which are unstable poles of  $K_0$ , extend admissible  $\hat{Q}$ 's to a class of unstable systems.  $\nabla$

A state-space realization of the generator of all stabilizing controllers in Lemma 2.1,  $J_0$ , can also be derived. To this end, bring in stabilizable and detectable realizations

$$P(s) = \left[ \begin{array}{c|c} A & B_u \\ \hline C_y & 0 \end{array} \right] \quad \text{and} \quad K_0(s) = \left[ \begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array} \right] \quad (3)$$

any pick any  $F_0$  and  $L_0$  such that  $A_0 + B_0 F_0$  and  $A_0 + L_0 C_0$  are Hurwitz. Coprime factors of  $K_0$  can then be constructed as in [16, Thm. 5.9], which eventually yields

$$J_0(s) = \left[ \begin{array}{ccc|cc} A_0 & 0 & 0 & B_0 & -L_0 \\ 0 & A_0 & B_0 C_y & 0 & -L_0 \\ 0 & B_u C_0 & A + B_u D_0 C_y & 0 & B_u \\ \hline C_0 & 0 & 0 & D_0 & I \\ -F_0 & F_0 & -C_y & I & 0 \end{array} \right] \quad (4)$$

$$= \left[ \begin{array}{c|c|c} A_0 & B_0 & -L_0 \\ \hline C_0 & D_0 & I \\ -F_0 & I & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ 0 & J_a(s) \end{array} \right]$$

with stable

$$J_a(s) := \left[ \begin{array}{cc|c} A_0 & B_0 C_y & -L_0 \\ \hline B_u C_0 & A + B_u D_0 C_y & B_u \\ F_0 & -C_y & 0 \end{array} \right].$$

The state dimension of  $J_0$  in (4) is in general higher than that of  $K_0$ . For instance, consider the static feedback case,  $K_0(s) = D_0$  for some  $D_0$  such that the matrix  $A + B_u D_0 C_y$  is Hurwitz. Then

$$J_0(s) = \left[ \begin{array}{c|cc} A + B_u D_0 C_y & 0 & B_u \\ \hline 0 & D_0 & I \\ -C_y & I & 0 \end{array} \right], \quad (4')$$

which is dynamic. In the observer-based case, where  $K_0(s) = -F(sI - A - B_u F - L C_y)^{-1} L$  for some  $F$  and  $L$  such that  $A + B_u F$  and  $A + L C_y$  are Hurwitz, the dimension of  $J_0$  is not increased. It can be verified that the choices  $F_0 = C_y$  and  $L_0 = -B_u$  result then in  $J_a = 0$  and the parametrization with

$$J_0(s) = \left[ \begin{array}{c|cc} A + B_u F + L C_y & -L & B_u \\ \hline F & 0 & I \\ -C_y & I & 0 \end{array} \right], \quad (4'')$$

as in [16, Thm. 12.8]. For a general  $K_0$ , we may aim at picking admissible  $F_0$  and  $L_0$  for which the order of  $J_a$  is minimal.

### B. Lifting technique

The idea of lifting is to convert analog signals to discrete sequences of functions operating over finite time intervals. Although mostly used to deal with systems with a constant sampling rate, see [2, Ch. 10] and the references therein, extensions of the technique to time-varying rates is effortless, at least at the level required in this paper.

Consider a sequence of time instances  $\{t_i\}_{i \in \mathbb{Z}^+}$  such that  $0 = t_0 < t_1 < t_2 < \dots$ . Then any analog signal  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$

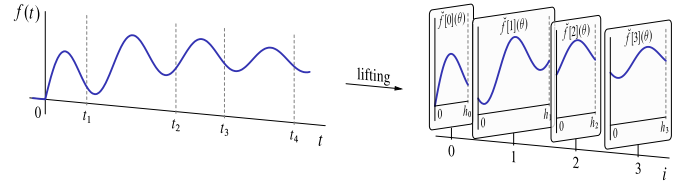


Fig. 2. Lifting transformation with nonuniform time axis partition

can be equivalently cast as a sequence of functions  $\{\check{f}[i]\}_{i \in \mathbb{Z}^+}$  such that  $\check{f}[i] : [0, h_i] \rightarrow \mathbb{R}^n$  is defined according to

$$\check{f}[i](\theta) = f(t_i + \theta), \quad i \in \mathbb{Z}^+, \theta \in [0, h_i]$$

where  $h_i := t_{i+1} - t_i$  is the length of the  $i$ th interval. The discrete sequence  $\{\check{f}[i]\}$  is said to be the *lifting* of the analog signal  $f(t)$  with respect to the  $t$ -axis partition by  $\{t_i\}$ . See Fig. 2 for a visualization of this transformation.

Any continuous-time system can then be lifted by lifting its input and output signals, resulting in a discrete-time system with infinite-dimensional input/output spaces. To be specific, consider a causal controller  $K : y \mapsto u$  described by the kernel representation

$$u(t) = \int_0^t k(t, \tau) y(\tau) d\tau \quad (5)$$

for an associated distribution  $k(t, \tau)$  (impulse response) such that  $k(t, \tau) = 0$  whenever  $t < \tau$ . The impulse response may be visualizing as shown in Fig. 3(a), where the unshaded area represents zero values. Relation (5) can be rewritten in the lifted domain as

$$\check{u}[i](\theta) = \sum_{j=0}^i \int_0^{h_j} k(t_i + \theta, t_j + \sigma) \check{y}[j](\sigma) d\sigma =: \left( \sum_{j=0}^i \check{K}_{ij} \check{y}[j] \right)(\theta)$$

This relation describes a discrete linear system, denote it  $\check{K}$ , whose kernel (impulse response)  $\check{K}_{ij}$  at each  $i, j$  is an integral operator mapping functions on  $[0, h_j]$  to functions on  $[0, h_i]$ . In terms of the kernel in Fig. 3(a), this transformation may be viewed as merely chopping the  $t$ - and  $\tau$ -axes into pieces according to  $\{t_i\}$ . The result, shown in Fig. 3(b), can then be thought of as a form of system matrix as in [2, Sec. 4.1].

The “diagonal” elements  $\check{K}_{ii}$  of a lifted impulse response are called its *feedthrough parts*. At each  $i \in \mathbb{Z}^+$  they are integral operators on  $[0, h_i]$  representing the direct connection between  $\check{y}[i]$  and  $\check{u}[i]$ . Given a lifted system  $\check{K}$ , by its *static part* we understand the lifted system, whose kernel is  $\check{K}_{ij} \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta. At each  $i$  this static part acts as  $\check{u}[i] = \check{K}_{ii} \check{y}[i]$ , which corresponds to the diagonal system

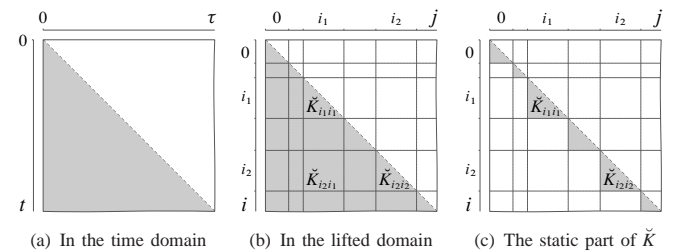


Fig. 3. Impulse responses of causal controllers



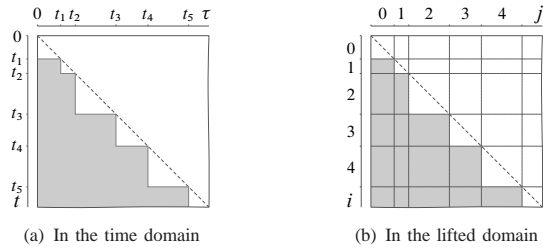


Fig. 4. Impulse responses of causal sampled-data controllers

matrix depicted in Fig.3(c). The following result, which is straightforward to verify, will be required in the sequel:

**Lemma 2.2:** Let  $G$  be an LTI system with the state-space realization  $(A, B, C, D)$  and let  $\check{G}$  be its lifting with respect to the time axis partition by  $\{t_i\}$ . Then the static part of  $\check{G}$  is the lifting of the continuous-time system  $\zeta \mapsto \xi$  verifying

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\zeta(t), & x(t_i) &= 0 \\ \xi(t) &= Cx(t) + D\zeta(t) \end{aligned}$$

for all  $t \in \mathbb{R}^+$  and  $i \in \mathbb{Z}^+$ .

### III. WHEN IS A CONTROLLER SAMPLED-DATA ?

The redesign approach of this paper hinges on converting analog controllers to sampled-data ones via *constraining* the former. The first step to this end is to understand how to characterize causal sampled-data controllers of the form in Fig. 1 among linear operators mapping the measurement signal  $y$  into the control signal  $u$ .

An important role in the reasonings below is played by the fact that the sampler and hold in Fig.1 are not fixed (and neither are the dimensions of the discrete signals  $\bar{y}$  and  $\bar{u}$ ). What should then be understood by a sampled-data controller? The picture appears to be easier to grasp via causality of the mapping  $y \mapsto u$ . Indeed, the very presence of the sampling operation inside the controller should imply that between two subsequent sampling instances  $u$  has no new information about  $y$ , no matter what A/D and D/A converters are used. In other words,  $u(t)$  for all  $t \in (t_i, t_{i+1})$  may be based on  $y(\tau)$  for  $\tau \leq t_i$  only. Any controller satisfying this causality constraint will be regarded as an *admissible one*. In terms of the kernel representation (5), admissibility then requires that

$$k(t, \tau) = 0 \text{ whenever } \tau > \max_{t_i \leq t} t_i. \quad (6)$$

This yields a staircase, instead of triangle, constraint on the impulse response, as shown in Fig.4(a).

Constraint (6) might not be convenient to incorporate into design procedures though, especially if the employed approach does not use the impulse response directly. The constraint, however, is substantially simplified if translated to the lifted domain associated with  $\{t_i\}$ . The stairs in Fig.4(a) fit then into the partition of the time axis, resulting in the system matrix in Fig.4(b). This suggests that (6) translates to the lifted domain as *strict causality*, i.e. the constraint that the feedthrough parts are zero. The following result, which may be viewed as an extension of [14, Thm. 1] to systems with

non-uniform sampling and without the finite dimensionality assumption about  $K$ , formalizes this observation:

**Theorem 3.1:** Let  $\check{K}$  be a linear system in the lifted domain with respect to the time axes partition by  $\{t_i\}$ .  $\check{K}$  is the lifting of a causal sampled-data system as in Fig. 1 with the sampling instances  $\{t_i\}$  iff  $\check{K}$  is strictly causal, i.e.  $\check{K}_{ii} = 0, \forall i \in \mathbb{Z}^+$ .

*Proof:* Follows by lifting (6).  $\square$

The strict causality is a more convenient system-theoretic notion to handle in various controller design approaches than constraint (6). This is the reason to introduce lifting.

### IV. STABILITY-PRESERVING REDESIGN

Consider an LTI plant  $P$ . Without loss of generality, assume that its transfer function  $P(s)$  is strictly proper (this simplifies technicalities but can be easily relaxed, see [16, p.454]). Let a causal LTI controller  $K_0$  internally stabilize<sup>1</sup>  $P$  and  $\{t_i\}_{i \in \mathbb{Z}^+}$  be a sequence of time instances such that

$$0 = t_0 < t_1 < \dots < t_i < \dots, \quad \text{with } \lim_{i \rightarrow \infty} t_i = \infty.$$

The problem studied in this section is to approximate  $K_0$  by a linear causal sampled-data controller with the sampling instances  $\{t_i\}$ , so that the closed-loop stability is preserved. By causal we understand a sampled-data controller as in Fig. 1, where  $\mathcal{S}$  produces discrete signals  $\bar{y}[i]$  at each  $t_i$  on the basis of measurements  $y(t)$  for  $t < t_i$ ,  $\check{K}$  is causal, and  $\mathcal{H}$  shapes the control signal  $u(t)$  in  $t \in [t_i, t_{i+1})$  on the basis of discrete signals  $\bar{u}[j]$  for  $j \leq i$ . We assume hereafter that the sampling instances  $t_i$  are not known a priori, but the length of the intersample intervals  $h_i := t_{i+1} - t_i$  is uniformly bounded.

#### A. Solution in the lifted domain

By Lemma 2.1,  $K_0$  generates the whole family of linear stabilizing controllers,  $K = \mathcal{F}_1(J_0, Q)$  for a given  $J_0$ , which is an augmentation of  $K_0$ , and arbitrary stable and causal  $Q$ . Clearly, any stabilizing sampled-data controller must belong to this family. It is therefore pertinent to understand, what conditions should be imposed on  $Q$  to produce sampled-data  $\mathcal{F}_1(J_0, Q)$ . The latter question, in turn, is convenient to address in the lifted domain, where a handy characterization of sampled-data controller exists, see Theorem 3.1.

In the lifted domain, the controller parametrization reads  $\check{K} = \mathcal{F}_1(\check{J}_0, \check{Q})$ , where  $\check{J}_0$  and  $\check{Q}$  are the lifted versions of  $J_0$  and  $Q$ , respectively, with an arbitrary stable  $\check{Q}$  such that its feedthrough terms  $\check{Q}_{ii}$  are causal. This LFT is then always well posed. Theorem 3.1 says that  $\check{K}$  is the lifting of a sampled-data system iff its feedthrough terms  $\check{K}_{ii} = 0$  for all  $i \in \mathbb{Z}^+$ . The feedthrough terms of  $\check{K}$  depend only on those of  $\check{J}_0$  and  $\check{Q}$  (because of their causality), i.e.  $\check{K}_{ii} = \mathcal{F}_1(\check{J}_{0,ii}, \check{Q}_{ii})$  for every  $i$ . Then, by [16, Lem. 10.4(c)],  $\check{Q}_{ii} = \mathcal{F}_u(\check{J}_{0,ii}^{-1}, \check{K}_{ii})$ . Hence, for every  $i$  we have that

$$\check{K}_{ii} = 0 \iff \check{Q}_{ii} = \check{Q}_{0,ii} := \begin{bmatrix} 0 & I \end{bmatrix} \check{J}_{0,ii}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

<sup>1</sup>The stability of a linear system  $G$  is understood throughout the paper as its boundedness as an operator  $L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ . In most cases the results remain unchanged if  $L^2(\mathbb{R}^+)$  is replaced with  $L^p(\mathbb{R}^+)$  for any  $p \geq 1$ .

This condition completely determines the feedthrough terms of  $\check{Q}$  and does not affect the rest of it, which is handy.

Two straightforward, yet nevertheless important, observations are in order here. First,  $\check{Q}_{0,ii}$  defined above is causal, because so is the continuous-time system  $J_0^{-1}$ . Second, the static lifted system  $\check{Q}_{\text{stat}}$ , whose impulse response operators

$$\check{Q}_{\text{stat},ij} = \begin{cases} \check{Q}_{0,ii} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

is stable, as it is the lifting of an LTI system whose state resets at every  $t_i$  with uniformly bounded<sup>2</sup>  $t_{i+1} - t_i$ . Consequently, any admissible  $\check{Q}$  can be presented as  $\check{Q} = \check{Q}_{\text{stat}} + \check{Q}_{\text{sd}}$  for a strictly causal  $\check{Q}_{\text{sd}}$ , which is thus the lifting of a sampled-data system, and  $\check{Q}$  is stable iff  $\check{Q}_{\text{sd}}$  is stable.

The discussion above can be summarized as follows:

**Lemma 4.1:** All causal stabilizing sampled-data controllers in the lifted domain can be parametrized as

$$\check{K}_{\text{sd}} = \mathcal{F}_1(\check{J}_0, \check{Q}_{\text{stat}} + \check{Q}_{\text{sd}})$$

for an arbitrary strictly causal stable  $\check{Q}_{\text{sd}}$ , where  $\check{Q}_{\text{stat}}$  is the static part of the (2, 2) sub-block of  $\check{J}_0^{-1}$ .

### B. Solution in the continuous-time domain

Although treating the problem in the lifted domain is simple conceptually, it does not result in a transparent solution. Our next step is thus to “peel off” the lifted-domain result of Lemma 4.1, i.e. to transform it back the time domain, where the structure of the resulting controllers is clear.

To this end, let

$$J_0(s) = \left[ \begin{array}{c|cc} A_J & B_{J1} & B_{J2} \\ \hline C_{J1} & D_0 & I \\ C_{J2} & I & 0 \end{array} \right],$$

(concrete expressions of the parameters of this realization in terms of realizations of  $P$  and  $K_0$  are given by (4)). The following theorem, which is a sampled-data version of the Youla-Kučera parametrization with unrestricted sampler and hold, is then the main result of this section:

**Theorem 4.2:** All causal stabilizing sampled-data controllers can be characterized as the interconnection of the sensor side “pre-processor”

$$\dot{x}_s(t) = A_J x_s(t) + B_{J1} y(t) + B_{J2} (u(t) - u_s(t)), \quad (7a)$$

where  $u_s = C_{J1} x_s + D_0 y$ , and the “post-processor”

$$\dot{x}_a(t) = (A_J - B_{J1} C_{J2}) x_a(t) + B_{J2} \eta(t) \quad (7b)$$

$$u(t) = (C_{J1} - D_0 C_{J2}) x_a(t) + \eta(t) \quad (7c)$$

at the actuation side, connected via their sampled states as

$$x_a(t_i) = x_s(t_i) \quad (7d)$$

and the signal  $\eta = Q_{\text{sd}}(C_{J2} x_s + y)$ , where  $Q_{\text{sd}}$  is an arbitrary causal and stable sampled-data system.

<sup>2</sup>The uniform boundedness is actually required only if the (2, 2) sub-block of  $J_0^{-1}$  is unstable. If this system is stable, which happens iff  $P$  is itself stable (cf. (2)), the result holds for any  $\{t_i\}$ .

*Proof:* The state-space realization of  $J_0^{-1}$  is obtained by [16, Lem. 3.15]. Using Lemma 2.2, we then end up with  $\check{Q}_{\text{stat}} : \check{\epsilon} \mapsto \check{\eta}_Q$  as the lifting of

$$\begin{aligned} \dot{x}_Q(t) &= A_J^\times x_Q(t) - B_{J12} \epsilon(t), & x_Q(t_i) &= 0 \\ \eta_Q(t) &= C_{J12} x_Q(t) - D_0 \epsilon(t) \end{aligned} \quad (8)$$

where  $A_J^\times := A_J - B_{J1} C_{J2} - B_{J2} C_{J1} + B_{J2} D_0 C_{J2}$ ,

$$B_{J12} := B_{J1} - B_{J2} D_0 \quad \text{and} \quad C_{J12} := C_{J1} - D_0 C_{J2}. \quad (9)$$

Denoting by  $\eta$  the output of  $Q_{\text{sd}}$  and by  $\epsilon$  the second output of  $J_0$ , the dynamics of  $J_0$  read

$$\begin{aligned} \dot{x}_J(t) &= A_J x_J(t) + B_{J1} y(t) + B_{J2} (\eta(t) + \eta_Q(t)) \\ u(t) &= C_{J1} x_J(t) + D_0 y(t) + \eta(t) + \eta_Q(t) \\ \epsilon(t) &= C_{J2} x_J(t) + y(t) \end{aligned}$$

(the second input of  $J_0$  is the sum of the outputs of  $Q_{\text{stat}}$  and  $Q_{\text{sd}}$ ). Combining this realization with (8), eliminating  $\eta_Q$ , and carrying out a state transformation yields (7) with  $x_s = x_J$  and  $x_a = x_J + x_Q$ .  $\square$

The signal  $u_s$  in pre-processor (7a) may be thought of as an emulation of the output of the analog controller  $K_0$ , which equals  $C_{J1} x_J + D_0 y$ . The pre-processor resembles then the state observer for  $J_0$ . The only difference is that the calculated output,  $u_s$ , is now compared with the actual control signal,  $u$ , produced by another system, via the sampling operation (7d).

The central controller, the one with  $Q_{\text{sd}} = 0$  (and  $\eta = 0$ ), can be presented in the form shown in Fig. 1. To describe its components, introduce the matrix functions

$$\begin{aligned} &\begin{bmatrix} \Lambda_{11}(\theta) & \Lambda_{12}(\theta) \\ 0 & \Lambda_{22}(\theta) \end{bmatrix} \\ &:= \exp \left( \begin{bmatrix} A_J - B_{J2} C_{J1} & B_{J2} C_{J12} \\ 0 & A_J - B_{J1} C_{J2} \end{bmatrix} \theta \right) \end{aligned}$$

with  $\Lambda_{11}(\theta) = e^{(A_J - B_{J2} C_{J1})\theta}$ ,  $\Lambda_{22}(\theta) = e^{(A_J - B_{J1} C_{J2})\theta}$ , and

$$\Lambda_{12}(\theta) = \int_0^\theta \Lambda_{11}(\theta - \sigma) B_{J2} C_{J12} \Lambda_{22}(\sigma) d\sigma \quad (10)$$

(by Van Loan’s formulae, see e.g. [2, Lem. 10.5.1]). Then:

**Corollary 4.3:** The “central” controller of Theorem 4.2 can be implemented as the sampled-data controller in Fig. 1 with the generalized sampler (A/D converter)  $\mathcal{S} : y \mapsto \bar{y}$

$$\bar{y}[i+1] = \int_0^{h_i} e^{(A_J - B_{J2} C_{J1})(h_i - \sigma)} B_{J12} y(t_i + \sigma) d\sigma, \quad (11a)$$

the discrete-time controller  $\bar{K} : \bar{y} \mapsto \bar{u}$

$$\bar{u}[i+1] = (\Lambda_{11}(h_i) + \Lambda_{12}(h_i)) \bar{u}[i] + \bar{y}[i+1], \quad (11b)$$

and the generalized hold (D/A converter)  $\mathcal{H} : \bar{u} \mapsto u$

$$u(t_i + \theta) = C_{J12} e^{(A_J - B_{J1} C_{J2})\theta} \bar{u}[i], \quad (11c)$$

where  $B_{J12}$  and  $C_{J12}$  are defined by (9).

*Proof:* Rewrite (7a) as

$$\dot{x}_s(t) = (A_J - B_{J2} C_{J1}) x_s(t) + B_{J12} y(t) + B_{J2} u(t),$$

so that

$$x_s(t_{i+1}) = \Lambda_{11}(h_i)x_s(t_i) + \int_0^{h_i} \Lambda_{11}(h_i - \sigma) \times (B_{J12}y(t_i + \sigma) + B_{J2}u(t_i + \sigma))d\sigma.$$

Now, (11a), the fact that  $u(t_i + \sigma) = C_{J12}\Lambda_{22}(\sigma)x_s(t_i)$ , which follows from (7b)–(7d) with  $\eta = 0$ , and (10) yield that

$$x_s(t_{i+1}) = (\Lambda_{11}(h_i) + \Lambda_{12}(h_i))x_s(t_i) + \bar{y}[i + 1].$$

The result follows by introducing  $\bar{u}[i] := x_s(t_i)$ .  $\square$

Controller (11) is well suited to networked implementation. Sampler (11a) requires uninterrupted access to the measured output  $y$  and should be implemented on the sensor side. Hold (11c) generates a complex waveform analog control signal  $u$ , so it should be implemented on the actuator side. The exchange of information between these parts, done via (11b), may be intermittent. It can be carried out either opportunistically, when network resources are available, or when menacing deviations from predicted behavior are detected. In any case, the nominal closed-loop system remains stable for any uniformly bounded sequence of sampling intervals  $\{h_i\}$ .

The control signal  $u$  generated by (7) is typically discontinuous because of jumps in  $x_a$  at  $t = t_i$ , cf. (7d). A workaround is to parametrize the set of analog stabilizing controllers in the form  $K = F_{lp}\mathcal{F}_1(\tilde{J}_0, Q)$  for some low-pass  $F_{lp}$ . This can be done by factoring  $K_0 = F_{lp}\tilde{K}_0$  and then applying Lemma 2.1 to  $\tilde{K}_0$  and the augmented plant  $\tilde{P} = PF_{lp}$ . In this case only  $\mathcal{F}_1(\tilde{J}_0, Q)$  is redesigned, so that the actual control signal is a filtered version of (7c). This factorization is sometimes a part of the design method, see Section VI for an example.

### C. Special cases

To illustrate the structure of the controller derived above, consider in this subsection some special cases. It is assumed throughout that the plant is given in terms of its state-space realization (3).

1) *Static  $K_0$* : Let  $K_0(s) = D_0$  for a  $D_0$  such that the matrix  $A + B_u D_0 C_y$  is Hurwitz. Then  $J_0(s)$  is given by (4') and (7) can be rewritten as

$$\dot{x}_s(t) = A x_s(t) + B_u u(t) - B_u D_0 (y(t) - C_y x_s(t)) \quad (12a)$$

$$\dot{x}_a(t) = A x_a(t) + B_u u(t), \quad x_a(t_i) = x_s(t_i) \quad (12b)$$

$$u(t) = D_0 C_y x_a(t) + \eta(t) \quad (12c)$$

The sensor-side part, (12a), is the standard full-order observer of the plant state with the gain  $L = B_u D_0$ . The actuator-side part, (12b)–(12c), mimics then the dynamics of the closed-loop system under the analog control law  $u = D_0 y + \eta$ .

2) *Observer-based  $K_0$* : In this case the generator of all stabilizing controllers,  $J_0$ , is given by (4''). Hence, (7a) reads

$$\dot{x}_s(t) = A x_s(t) + B_u u(t) - L(y(t) - C_y x_s(t)), \quad (13a)$$

which is again an observer, and (7b)–(7d) read

$$\dot{x}_a(t) = A x_a(t) + B_u u(t), \quad x_a(t_i) = x_s(t_i) \quad (13b)$$

$$u(t) = F x_a(t) + \eta(t). \quad (13c)$$

In the intermittent sampling case, this controller structure was proposed in [11], although with no stability proof. Apparently, the first proof of the closed-loop stability under this scheme was offered in [21]. In the constant  $h_i$  case, earlier proofs exist. If presented in form (11), this is exactly the optimal controller configuration of [7, Thm. 5.1]. The even earlier result of [6, Thm. 3.1] is also essentially the same system, sans the absorption of  $Q_{\text{stat}}$  into  $J_0$ . See also [9, Ch. 3] for an analysis of the same controller under the constant sampling rate and parametric plant uncertainty.

Curiously, the redesigned static controller (12) is a special case of the redesigned observer-based controller (13), under  $L = B_u D_0$  and  $F = D_u C_y$ . Consequently, the use of static controllers offers no advantage over observer-based controllers in terms of simplicity for the proposed redesign procedure.

### D. Complexity reduction via $Q_{sd}$

The freedom in the choice of  $Q_{sd}$  can be used to reduce the complexity of the controller of Theorem 4.2. Consider, for example, the following  $Q_{sd} : y - C_{J2}x_s \mapsto \eta$ :

$$\begin{aligned} \dot{x}_\eta(t) &= A_\eta x_\eta(t), \quad x_\eta(t_i) = B_\eta(y(t_i) - C_{J2}x_s(t_i)) \\ \eta(t) &= C_\eta x_\eta(t) \end{aligned} \quad (14)$$

which is the cascade of the ideal sampler and a generalized hold as in (11c), just with different parameters. System (14) is stable for any  $A_\eta$ ,  $B_\eta$ , and  $C_\eta$ , because it resets at every  $t_i$ . With this choice, the actuation-side dynamics (7b)–(7c) read

$$\begin{aligned} \begin{bmatrix} \dot{x}_a(t) \\ \dot{x}_\eta(t) \end{bmatrix} &= \begin{bmatrix} A_J - B_{J1}C_{J2} & B_{J2}C_\eta \\ 0 & A_\eta \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_\eta(t) \end{bmatrix} \\ u(t) &= \begin{bmatrix} C_{J1} - D_0C_{J2} & C_\eta \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_\eta(t) \end{bmatrix} \end{aligned}$$

with the following effect of (7a) on them:

$$\begin{bmatrix} x_a(t_i) \\ x_\eta(t_i) \end{bmatrix} = \begin{bmatrix} I \\ -B_\eta C_{J2} \end{bmatrix} x_s(t_i) + \begin{bmatrix} 0 \\ B_\eta \end{bmatrix} y(t_i).$$

If  $C_\eta = C_{J1} - D_0C_{J2}$ , then  $u$  depends only on  $\tilde{x}_a := x_a + x_\eta$ . If then  $A_\eta = A_J^\times$  defined after (8), the signal  $\tilde{x}_a$  becomes independent of  $x_\eta$  (can be seen by a similarity transformation). As a result, we end up with essentially unchanged actuator-end equations (just with  $\eta = 0$ ) and with the new interconnection

$$x_a(t_i) = (I - B_\eta C_{J2})x_s(t_i) + B_\eta y(t_i). \quad (7d')$$

in place of (7d). We may then seek for  $B_\eta$  that renders some modes of (7a), which are the eigenvalues of  $A_J - B_{J2}C_{J1}$ , unobservable through  $I - B_\eta C_{J2}$ . Unobservable dynamics may then be safely canceled, reducing the order of (7a).

A possible procedure for carrying out such a reduction is as follows. Assume w.l.o.g. that  $C_{J2}$  has full row rank. Let  $V_2$  be a matrix such that  $\text{Im } V_2$  is  $(A_J - B_{J2}C_{J1})$ -invariant and  $C_{J2}V_2$  is left invertible. Pick  $B_\eta$  as any solution of  $B_\eta C_{J2}V_2 = V_2$ . In this case  $\text{Im } V_2 = \ker(I - B_\eta C_{J2})$ , which implies that  $\text{Im } V_2$  is the unobservable subspace of the  $(I - B_\eta C_{J2}, A_J - B_{J2}C_{J1})$ . Hence, all modes of  $A_J - B_{J2}C_{J1}|_{\text{Im } V_2}$  are unobservable through  $I - B_\eta C_{J2}$  and can thus be canceled. The maximal reduction is attained if there is an admissible  $V_2$  such that  $C_{J2}V_2$  is square.

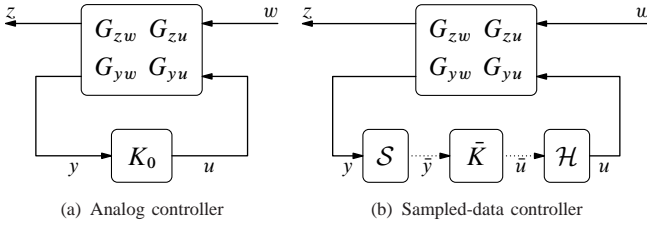


Fig. 5. Standard problems

The choice of  $B_\eta$  is particularly simple in the static state-feedback case, which corresponds to (12) with  $C_y = I$  and  $D_0 = F$  for some  $F$  such that  $A + B_u F$  is Hurwitz. With the choice  $B_\eta = I$ , equation (7d') reads  $\tilde{x}_a(t_i) = x(t_i)$ , which renders observer (12a) redundant. This yields the control law

$$u(t) = F e^{(A+B_u F)(t-t_i)} x(t_i), \quad \forall t \in [t_i, t_{i+1})$$

which effectively reproduces the algorithm of [22] (see also [9, Ch. 5]) and [23] (the latter also adds the effect of a piecewise constant disturbance estimate to the generated  $u$ ).

## V. PERFORMANCE-GUARANTEEING REDESIGN

The procedure of Section IV produces a family of stabilizing sampled-data controllers from a given analog controller  $K_0$ . Of this family one would naturally prefer a controller that is close to  $K_0$ , in whatever sense. This section studies situations when the closeness between  $K_0$  and its sampled-data approximation is measured in terms of the attained closed-loop performance.

To this end, the setup is extended to the so-called “standard problem” of the form depicted in Fig. 5(a). The performance of this system is quantified by a norm, either  $H^2$  or  $H^\infty$ , of the closed-loop system  $T_{zw} := \mathcal{F}_1(G, K_0)$  from  $w$  to  $z$ . It is assumed that  $K_0$  guarantees certain performance level and the goal is to find a sampled-data controller that can deliver a comparable performance level for the setup in Fig. 5(b).

*Remark 5.1 (viewpoint):* The problems addressed in this section might also be viewed as merely the design of (sub)optimal sampled-data controllers for intermittent sampling. But optimality might make little engineering sense per se. Rather, it is a powerful tool to design “good” analog controllers. For that reason, solving the very same optimization problem for a sampled-data controller is treated here as a tool of redesigning a chosen analog controller  $K_0$ .  $\nabla$

Throughout this section, we assume that

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right]$$

and that the standard assumptions [16, p. 384] are satisfied (including the normalizations  $D'_{zu} D_{zu} = I$  and  $D_{yw} D'_{yw} = I$ ). The solution procedure is again to start with a parametrization, now of all suboptimal analog controllers, and then seek for a “least harmful”  $Q$ -parameter for which the resulting controller is a sampled-data one.

### A. $H^2$ performance

Let  $K_0$  be the  $H^2$ -optimal controller for the problem in Fig. 5(a) and  $\{t_i\}$  be a sequence of sampling instances. The problem studied below is to find the optimal sampled-data controller, of the form depicted in Fig. 1, for the same generalized plant.

The  $H^2$  norm of a linear system can be roughly viewed as the  $L^2(\mathbb{R}^+)$ -norm of its impulse response. In the LTI case, it is sufficient to consider the response to the impulse applied at  $t = 0$ , which leads to the conventional definition [16, p. 98]. The response of time-varying systems to impulses applied at different time instances might differ. A way to generalize the notion of the  $H^2$  norm to such systems is via averaging. Namely, let  $G$  be a linear system described by (5). Then we may define (see e.g. [24] or [25, §2.1.2]) its  $H^2$  norm as

$$\|G\|_2^2 := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^\infty \|g(t, \tau)\|_F^2 d\tau dt, \quad (15)$$

where  $\|\cdot\|_F$  denotes the Frobenius matrix norm. This quantity may also be thought of as the average output variance if the input is a zero mean white noise process. In general, (15) is a semi-norm, although in some special cases, like periodic systems, it is a norm. It reduces to the standard definition if  $G$  is time invariant.

The main result of this sub-section is formulated below:

*Theorem 5.1:* Let the analog  $H^2$  problem associated with the system in Fig. 5(a) be well posed and  $F$  and  $L$  be the state-feedback and filter gains associated with this problem. Then the optimal  $H^2$  performance attainable by sampled-data controllers for a given sequence of sampling instances  $\{t_i\}$  is

$$\gamma_{\{t_i\}}^2 = \gamma_0^2 + \lim_{i \rightarrow \infty} \frac{1}{t_i} \sum_{j=0}^{i-1} \int_0^{h_j} \int_0^{h_j - \tau} \|F e^{A\tau} L\|_F^2 d\tau dt,$$

where  $\gamma_0$  is the optimal  $H^2$  performance attainable by analog controllers. The  $H^2$  performance attained by the sampled-data controller given by (13) with  $\eta = Q_{sd}(y - C_y x_s)$  is then  $\|T_{zw}\|_2^2 = \gamma_{\{t_i\}}^2 + \|Q_{sd}\|_2^2$ .

*Proof:* See Appendix.  $\square$

Note that the optimal sampled-data controller is not unique. Because (15) is a semi-norm, there are nonzero  $Q_{sd}$  such that  $\|Q_{sd}\|_2 = 0$ . Any such  $Q_{sd}$  produces an optimal controller.

An intriguing question is under what sampling pattern  $\{t_i\}$  the attainable performance is minimal. Of course, this question makes sense only if the “average” sampling period is fixed. Another assumption that should be made in this respect is that the sampling pattern is periodic. Otherwise, an alternation of any finite number of sampling instances  $t_i$  has no effect on  $\gamma_{\{t_i\}}$ . Thus, assume that there is an  $N$  such that  $h_{i+N} = h_i$  for all  $i \in \mathbb{Z}^+$  and that

$$h_{av} := \frac{1}{N} \sum_{i=0}^{N-1} h_i = \frac{t_N}{N} \quad (16)$$

is fixed. In this case

$$\gamma_{\{t_i\}}^2 = \gamma_0^2 + \frac{1}{N h_{av}} \sum_{j=0}^{N-1} \int_0^{h_j} \int_0^{h_j - \tau} \|F e^{A\tau} L\|_F^2 d\tau dt$$



(and, as a matter of fact, the optimal  $Q_{sd} = 0$  is unique now). The optimal sampling pattern is then given as follows:

*Proposition 5.2:* If  $K_0 \neq 0$ , the unique optimal sampling pattern for a fixed  $h_{av}$  in (16) and any  $N \in \mathbb{Z}^+ \setminus \{0\}$  is the uniform sampling, i.e.  $h_i = h_{av}$  for all  $i \in \mathbb{Z}^+$ .

*Proof:* First,  $K_0(s) = -F(sI - A - B_u F - L C_y)^{-1} L = 0$  iff  $F(sI - A)^{-1} L = 0$ , which is readily verified via the Kalman canonical decomposition [16, Thm. 3.10]. Hence, the condition of the proposition guarantees that  $F e^{At} L \not\equiv 0$  in any finite interval of  $\mathbb{R}^+$ .

Let us start with the case of  $N = 2$ . Sampling periods can then be parametrized as  $h_0 = h - \delta$  and  $h_1 = h + \delta$  for  $\delta \in [-h, h]$  and the optimal performance is

$$\gamma_{\{t_i\}}^2 = \gamma_0^2 + \frac{\gamma_1(h + \delta) + \gamma_1(h - \delta)}{2h},$$

where

$$\gamma_1(h) := \int_0^h \int_0^{h-\tau} \|F e^{At} L\|_F^2 dt d\tau.$$

It can be verified, using the Leibniz integral rule, that

$$\frac{d\gamma_1(h + \delta)}{d\delta} = \int_0^{h+\delta} \|F e^{At} L\|_F^2 dt,$$

so that

$$\frac{d\gamma_{\{t_i\}}^2}{d\delta} = \frac{1}{2h} \int_{h-\delta}^{h+\delta} \|F e^{At} L\|_F^2 dt$$

has the same sign as  $\delta$  and is zero iff  $\delta = 0$ . This proves the statement of the Proposition.

Now consider the case of  $N > 2$ . If not all  $h_i$  are equal, we can always find a  $j > 1$  such that  $h_{j-1} \neq h_j$ . The replacement of  $t_j$  with  $(t_{j+1} + t_{j-1})/2$  then decreases  $\gamma_1(h_{j-1}) + \gamma_1(h_j)$  and affects no other  $\gamma_1(h_i)$ . Hence, there always a pattern yielding a better performance. This procedure fails to reduce  $\gamma_{\{t_i\}}$  only if all  $h_i = h_{av}$ , which completes the proof.  $\square$

Proposition 5.2, which establishes that the uniform sampling is advantageous, appears to disagree with some earlier results. This aspect is clarified in the following two remarks.

*Remark 5.2 (alternative choices of the  $H^2$  norm):* A variable sampling rate scheme to improve the LQR performance in sampled-data systems was proposed in [26]. It is based on the rate of change of the optimal analog control signal and is optimal for 1-order systems. The problem studied in [26] is different from that studied here though. First, it assumes the zero-order hold and the ideal sampler. This is different, and more restrictive, from the setup with free hold and sampler. Second, and most importantly, the performance measure considered in [26] is different. The LQR optimization effectively minimizes the energy of the response to the impulse applied at  $t = 0$  only. In other words, it does not involve averaging. As follows from the proof of Theorem 5.1, if this philosophy were used in the  $H^2$  design for the system in Fig. 5(a), the optimal performance would be

$$\|T_{zw}\|_2^2 = \gamma_0^2 + \int_0^{h_0} \|F e^{At} L\|_F^2 dt.$$

The obvious choice is then  $t_1 \rightarrow 0$ , which recovers the analog performance irrespective of the other sampling instances. But

this design would make no practical sense. Another possibility, something between (15) and LQR, would be to consider

$$\|G\|_2^2 := \lim_{i \rightarrow \infty} \frac{1}{i} \sum_{j=0}^{i-1} \int_{t_j}^{\infty} \|g(t, t_j)\|_F^2 dt.$$

Consider what happens with this choice when the sampling pattern is 2-periodic. In that case,

$$\|T_{zw}\|_2^2 = \gamma_0^2 + \frac{1}{2} \left( \int_0^{h+\delta} \|F e^{At} L\|_F^2 dt + \int_0^{h-\delta} \|F e^{At} L\|_F^2 dt \right)$$

so that

$$\frac{d\|T_{zw}\|_2^2}{d\delta} = \frac{\|F e^{A(h+\delta)} L\|_F^2 - \|F e^{A(h-\delta)} L\|_F^2}{2}.$$

Similarly to the proof of Proposition 5.2, this function equals zero at  $\delta = 0$ . But this might neither be the only such point nor the point of the local minimum, depending on the parameters. For example, assume that the system is 1-order, i.e.  $A$ ,  $F$ , and  $L$  are scalars. In this case, the sign of the derivative of the optimal performance equals  $\text{sign}(e^{A\delta} - e^{-A\delta})$ . Thus, if the system is unstable ( $A > 0$ ), the uniform sampling is still the best option. But if the system is stable ( $A < 0$ ), the uniform sampling is the worst scenario and the best option is to alternate short and long sampling intervals. If  $A = 0$ , the sampling pattern is irrelevant. If  $G$  has higher order dynamics, the optimal sampling pattern might be more complicated.  $\nabla$

*Remark 5.3 (realization vs. process):* Another way to assign the sampling pattern is to use event-based mechanisms [8, 10]. Some results of this kind analyze the  $H^2$  performance. For example, the Lebesgue sampling strategy of [27] (see also [8, Sec. 3]) may result in a significant relaxation of the average sampling rate (by a factor of 3 in the case where  $A = D_\bullet = 0$  and  $B_\bullet = C_\bullet = 1$ ). The cause of this improvement may lie in the ability of event-based sampling to make use of the information about the effect of a *particular realization* of  $w$  on the system, rather than treating  $w$  as a *random process*. It may be interesting in this respect to investigate the possibility to use the signal  $Q_{\text{stat}}(y - C_y x_s)$ , with  $Q_{\text{stat}}$  as in (20), as the basis for event generation. This would be qualitatively different from existing event generation mechanisms as it involves low-pass filtering of the estimation error. This element may be useful in avoiding Zeno behavior [10] and may lead to performance-justified switching, see the example in §VI-B.  $\nabla$

## B. $H^\infty$ performance

Unlike the  $H^2$  case, the  $H^\infty$  performance measure admits a clean and unambiguous generalization to time-varying systems, as the  $L^2(\mathbb{R}^+)$  induced norm. Denote by  $\gamma_{\text{opt}} \geq 0$  the optimal  $H^\infty$  performance attainable for the standard problem associated with Fig. 5(a) by an analog controller. Let  $K_0$  be the central  $\gamma$ -suboptimal controller for a  $\gamma > \gamma_{\text{opt}}$ . This  $K_0$  generates the whole family of  $\gamma$ -suboptimal controllers. The question asked below is under what conditions on the sequence of sampling instances  $\{t_i\}$  this family contains a sampled-data controller of the form depicted in Fig. 1.



To formulate the result, we need the Riccati equations

$$\begin{aligned} XA + A'X + C_z' C_z + \gamma^{-2} X B_w B_w' X - F' F &= 0, \\ AY + YA' + B_w B_w' + \gamma^{-2} Y C_z' C_z Y - LL' &= 0, \end{aligned}$$

where  $F := -B_u' X - D_{zu}' C_z$  and  $L := -Y C_y' - B_w D_{yw}'$ . The solutions  $X$  and  $Y$  are called stabilizing if the matrices  $A_F := A + \gamma^{-2} B_w B_w' X + B_u F$  and  $A_L := A + \gamma^{-2} Y C_z' C_z + L C_y$  are Hurwitz. It is known [16, Thm. 16.4] that  $\gamma > \gamma_{\text{opt}}$  iff the stabilizing solutions exist and are such that  $X \geq 0$ ,  $Y \geq 0$ , and  $\rho(YX) < \gamma^2$ . We then have:

**Theorem 5.3:** Let  $\gamma > \gamma_{\text{opt}}$ . Then there is a  $\gamma$ -suboptimal sampled-data controller for a given sequence of sampling instances  $\{t_i\}$  iff there exists a solution to the differential Riccati equation

$$\begin{aligned} \dot{P}(t) &= AP(t) + P(t)A' \\ &\quad + B_w B_w' + \gamma^{-2} P(t) C_z' C_z P(t), \quad P(0) = Y \end{aligned}$$

such that  $\rho(P(t)X) < \gamma^2$ ,  $\forall t \in [0, h_i]$  and every  $i \in \mathbb{Z}^+$ . If the condition holds, a  $\gamma$ -suboptimal sampled-data controller is

$$\dot{x}_s(t) = A_L x_s(t) - Ly(t) + (B_u + \gamma^{-2} Y C_z' D_{zu})u(t), \quad (17a)$$

$$\dot{x}_a(t) = A_F x_a(t), \quad x_a(t_i) = (I - \gamma^{-2} YX)^{-1} x_s(t_i) \quad (17b)$$

$$u(t) = Fx_a(t). \quad (17c)$$

*Proof:* See Appendix.  $\square$

**Remark 5.4 (closed-loop stability):** The stability of the closed-loop system under the control law (17) is guaranteed only if the condition of Theorem 5.3 holds for all  $h_i$ . This is in contrast to the  $H^2$  case, where the controller is stabilizing even if it does not guarantee a required performance level.  $\nabla$

**Remark 5.5 (generating disturbances):** In terms of  $\tilde{x}_s := (I - \gamma^{-2} YX)^{-1} x_s$  the sensor-side dynamics in (17a) read

$$\begin{aligned} \dot{\tilde{x}}_s(t) &= A \tilde{x}_s(t) + B_w \tilde{w}_\gamma(t) + B_u u(t) \\ &\quad - \tilde{L}(y(t) - C_y \tilde{x}_s(t) - D_{yw} \tilde{w}_\gamma(t)), \end{aligned}$$

where  $\tilde{L} := (I - \gamma^{-2} YX)^{-1} L$  and  $\tilde{w}_\gamma := \gamma^{-2} B_w' X \tilde{x}_s$ . This is the  $H^\infty$  estimator for the analog control signal  $u = Fx$  in the presence of the “worst-case” disturbance  $w_\gamma = \gamma^{-2} B_w' Xx$ , where  $x$  is the state of  $G$ , see [16, Sec. 16.8]. In other words, controller (17) generates the disturbance under the worst-case scenario for its analog prototype. This is different from the strategy proposed in [23], where the sampled-data controller uses a piecewise-constant disturbance that “explains” the last deviation of the measured state from the calculated one.  $\nabla$

Some more observations are in order. The solvability condition of Theorem 5.3 holds for every  $\gamma > \gamma_{\text{opt}}$  provided  $\sup_i h_i$  is sufficiently small. As  $\gamma \rightarrow \infty$ , controller (17) recovers the  $H^2$ -optimal controller of Theorem 5.1. If transformed to the form of Corollary 4.3, controller (17) coincides with the  $H^\infty$  controller in [7, Thm. 5.2], modulo replacing the sampling instances  $ih$  with arbitrary  $t_i$ . The worst-case performance is determined by the longest sampling interval, which is non-obvious for time-varying sampled-data systems in general.

Apropos of the worst-case sampling, the following result, whose proof is straightforward, may be thought of as the  $H^\infty$  counterpart of Proposition 5.2:

**Proposition 5.4:** Let  $h_\gamma$  be the least upper bound for  $h_i$  that satisfy the solvability condition of Theorem 5.3 for a given  $\gamma$ . Then the periodic sampling with the sampling period  $h_\gamma$  has the slowest average sampling rate among all sampling patterns for which the  $H^\infty$  performance level of  $\gamma$  is attainable.

## VI. EXAMPLE: DESIGN VIA $H^\infty$ LOOP SHAPING

This section considers a numerical example, whose purpose is twofold: to illustrate the proposed approach and to show its application to the  $H^\infty$  loop shaping method of McFarlane and Glover [15], which requires some light adjustments.

### A. Intermittent redesign for $H^\infty$ loop shaping

The  $H^\infty$  loop shaping is a design procedure that uses the classical loop shaping guidelines for choosing weights and casts the phase shaping around the crossover, the “far from the critical point” requirement in the classical control, as a robust stability problem. Each iteration of this method consists of two steps. First, weighting functions  $W_o$  and  $W_i$  are chosen to shape the magnitude (singular values) of  $P_{\text{msh}} = W_o P W_i$ . This step is technically simple and aims at shaping loop gains in the low- and high-frequency ranges. Second, a special robust stability problem is solved for  $P_{\text{msh}}$  to render the closed-loop system stable and as far from the stability margin as possible. The choice of the robustness setup in this step is meaningful. It is the robustness to unstructured  $H^\infty$  uncertainties in the normalized coprime factors of  $P_{\text{msh}}$ . Although normally not related to the plant physics, this problem has two important advantages: its solution is non-iterative and it equally penalizes all four closed-loop frequency responses (see [15, §4.5.1]). The latter means that cancellations of stable lightly damped poles/zeros are not encouraged, in contrast to some other optimization-based settings, like the weighted/mixed sensitivity. If a satisfactory loop  $P_{\text{msh}} K_0$  is reached with some choice of  $W_o$  and  $W_i$  by an  $H^\infty$  (sub)optimal controller  $K_0$ , the resulting controller for the original plant is  $K = W_i K_0 W_o$ .

The robust stability problem solved in the second step is an  $H^\infty$  optimization problem, whose attainable performance level may serve as a success indicator [15, Sec. 6.4]. This renders the redesign problem of §V-B well suited for this method. We actually only need to redesign  $K_0$ , the addition of the weights, which are in the series connection with  $K_0$ , does not change the sampled-data nature of the controller. Indeed, the series of causal and strictly causal systems in the lifted domain is always strictly causal, see [14, §5.3] for details.

Assume that  $P_{\text{msh}}(s) = C(sI - A)^{-1}B$ . The optimal attainable analog performance for the  $H^\infty$  problem solved during the loop shaping iterations is  $\gamma_{\text{opt}} = \sqrt{1 + \rho(YX)}$ , where  $X \geq 0$  and  $Y \geq 0$  are the stabilizing solutions to the Riccati equations (in fact,  $H^2$  Riccati equations)

$$\begin{aligned} A'X + XA + C'C - XBB'X &= 0, \\ AY + YA' + BB' - YC'CY &= 0. \end{aligned}$$

The parametrization of all  $\gamma$ -suboptimal solutions can then be

parametrized [15, Thm. 4.14] as  $\mathcal{F}_1(J_\gamma, Q)$ , where

$$J_\gamma(s) = \left[ \begin{array}{c|cc} A - BB'X - Z_\gamma YC' & Z_\gamma YC' & Z_\gamma B \\ \hline -B'X & 0 & I \\ -C & I & 0 \end{array} \right] \quad (18)$$

and  $Q$  is any linear system whose  $L^2(\mathbb{R}^+)$ -induced norm  $\|Q\| < \sqrt{\gamma^2 - 1}$ . Here  $Z_\gamma := ((1 - \gamma^{-2})I - \gamma^{-2}YX)^{-1} > I$  is well defined for every  $\gamma > \gamma_{\text{opt}}$ . The following corollary of Theorem 5.3 can then be formulated:

*Corollary 6.1:* Let  $\gamma > \sqrt{1 + \rho(YX)}$ . Then there is a  $\gamma$ -suboptimal sampled-data controller for a given sequence of sampling instances  $\{t_i\}$  iff there exists a solution to the differential Riccati equation

$$\begin{aligned} \dot{P}(t) = & (A - YC'C)P(t) + P(t)(A' - C'CY) \\ & + BB' + \frac{1}{1-\gamma^2}P(t)C'CP(t), \quad P(0) = Y \end{aligned}$$

such that  $\rho(P(t)X) < \gamma^2 - 1$ ,  $\forall t \in [0, h_i]$  and every  $i \in \mathbb{Z}^+$ . If this condition holds, a sampled-data controller guaranteeing the same robustness level as that under  $K_0$  is

$$\dot{x}_s(t) = Ax_s(t) + Bu(t) + YC'(y(t) - Cx_s(t)), \quad (19a)$$

$$\dot{x}_a(t) = Ax_a(t) + Bu(t), \quad x_a(t_i) = Z_\gamma x_s(t_i) \quad (19b)$$

$$u(t) = -B'Xx_a(t). \quad (19c)$$

*Proof:* Follows by the same steps as the proof of Theorem 5.3.  $\square$

Curiously,  $Z_\gamma$  in (19b) is the only parameter of the controller that depends on  $\gamma$ . It may be of interest to investigate the possibility to adjust  $Z_\gamma$  on-line.

### B. Dampening a pendulum

Consider the problem of controlling a pendulum, which is mounted on a cart driven by a DC motor. The system has one input (the motor voltage) and two regulated outputs (the cart position and the pendulum angle). Assume that the controller comprises two loops. An internal servo loop, which is given and implemented as a 1DOF unity-feedback system, controls the cart position. Our goal is to design the external loop, which aims at dampening pendulum oscillations during command response of the cart. The external loop measures the pendulum angle and modifies the reference signal to the inner loop. This way, the reference signal for the cart is treated as the load disturbance against which the external loop acts.

Let the transfer function from the servo reference signal to the pendulum angle be

$$P(s) = -\frac{42s^2}{(s+18)(s^2+0.02s+23)}.$$

It has a pair of lightly damped poles at  $s = -0.01 \pm j4.796$ , so the control goal is to dampen them by feedback. To this end, we design an analog controller via the  $H^\infty$  loop shaping procedure. The choice

$$W_1(s) = \frac{5}{s+2} \quad \text{and} \quad W_0(s) = 1$$

yields a satisfactory loop with low  $\gamma_{\text{opt}} = 1.7213$ . Consider then the design with  $\gamma = 3.703 \approx 2.151\gamma_{\text{opt}}$  (the rationale

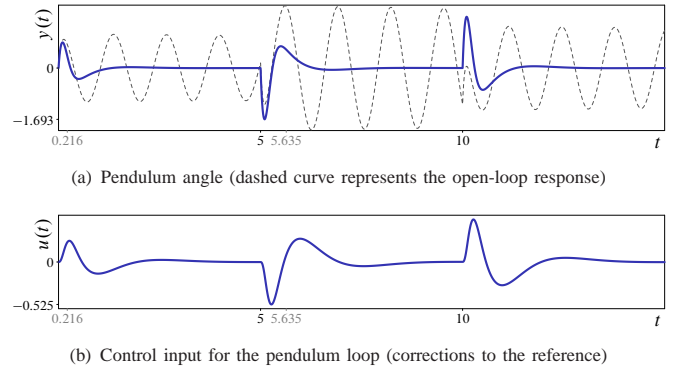


Fig. 6. Responses to a square wave, analog  $K_0$  designed for  $\gamma = 3.703$

behind this choice will be clarified later on), which produces the central analog controller

$$K_0(s) = W_i(s) \frac{12.534(s+18.85)(s+1.839)(s+0.2895)}{(s^2+1.91s+1.514)(s^2+37.26s+547.4)}.$$

The response of the resulted closed-loop system to a square wave load disturbance with a magnitude of  $\pm 0.5$  and a period of 10 sec, is shown in Fig. 6 by solid blue lines. Dampening properties the designed feedback are apparent from comparing the closed-loop output response to that of the open-loop plant (dashed line in Fig. 6(a)).

To redesign  $K_0$ , consider first how the condition of Corollary 6.1 on  $\{t_i\}$  depend on the robustness level  $\gamma$ . Calculating the least upper bound on the admissible sampling period at each  $\gamma > \gamma_{\text{opt}}$ , we end up with the plot in Fig. 7. Expectably,

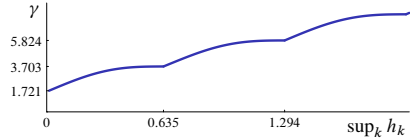


Fig. 7. Attainable  $\gamma$  as a function of the largest sampling interval

the required  $\sup_i h_i$  for  $\gamma$ 's close to  $\gamma_{\text{opt}}$  is quite close to zero, which leaves little room for investigating properties of intermittent sampling. It therefore makes sense to consider larger  $\gamma$ . The value chosen in the design of  $K_0$  is at the point where the slope of the curve in Fig. 7 is zero (so minimal damage for the increase of  $h_i$ ). The maximal admissible sampling period in this case is 0.635, which is rather slow from the classical sampled-data control viewpoint, as the corresponding Nyquist frequency of almost 5 rad/sec is comparable with the largest loop crossover of 7.75 rad/sec, see also the transients in Fig. 6.

Having the bound for admissible sampling rates and complete freedom in choosing the sampling pattern within this bound, let us dream up the following strategy for the choice of  $t_i$ . Consider the signal  $\eta = Q_{\text{stat}}(y - Cx_s)$ , where  $Q_{\text{stat}}$  is given by (22), adopted to  $J_\gamma$  in (18). This signal is reset at every sampling instance  $t_i$ . As the norm of this  $Q_{\text{stat}}$  determines the  $H^\infty$  performance, we may use the  $L^2$ -norm of  $\eta$  as a basis for event generation. To this end, let  $\theta_i$  be the solution of

$$\int_0^{\theta} \eta'(t_i + t)\eta(t_i + t)dt = 0.025^2$$

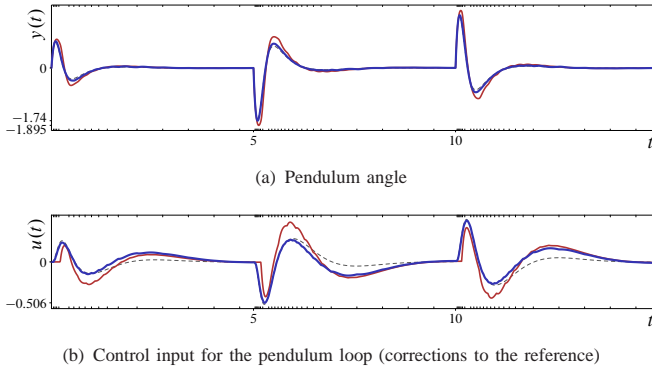


Fig. 8. Responses to a square wave, intermittent redesign of  $K_0$  (blue lines: event-based sampling, marked as the x-axis ticks; red lines: uniform sampling with the same density; gray dashed lines: analog controller)

and consider the following sampling generation mechanism:

$$h_i = \min\{\theta_i, 0.635\},$$

which is easy to implement. In other words, the controller samples either as the  $L^2$  norm of  $\eta$  reaches 0.025 or after 0.635 sec if the norm does not reach this level by then.

Simulation results with this controller are presented in Fig. 8 by blue lines. The resulted sampling instances are marked as the x-axis ticks. Intuitively, the sampling rate increases during the transients and decreases as the steady state is reached. One can see that the output response is quite close to the response under the analog  $K_0$  (dashed gray line in Fig. 8(a)). This is noteworthy, taking into account that the average sampling period here,  $h_{av} = 0.216$ , is still rather slow (the corresponding Nyquist frequency, 14.5 rad/sec, exceeds the largest crossover of the analog loop only by a factor of 2). For the sake of comparison, the red lines in Fig. 8 present responses under periodic sampling with  $h_i = 0.216, \forall i$ . Note that the control signals  $u(t)$  are continuous functions under both sampling strategies. This is because the discontinuous signal generated by (19c) is then filtered by the low-pass  $W_i$ . A larger pole excess in  $W_i(s)$  would result in a differentiable  $u(t)$ .

## VII. CONCLUDING REMARKS

The paper has studied the problem of digital redesign of analog controllers under intermittent, possibly unknown a priori, sampling. The main idea, borrowed from [14], is to use the characterization of causal sampled-data controllers as the set of all strictly causal systems in the lifted domain to extract sampled-data controllers from Youla-like parametrizations of satisfactory analog controllers. The resulting controllers are always stabilizing and, if optimal control parametrizations are considered, performance guaranteeing. As a byproduct of the proposed approach, the  $H^2$  and  $H^\infty$  problems under intermittent sampling have been solved. In both cases the (sub)optimal control laws are explicit and readily computable. It has also been proved that the uniform sampling is optimal among all sampling patterns with a given sampling density.

Some extensions of the results put forward in this paper should be immediate. For example, adding a single loop delay can be addressed via the loop shifting approach, similarly

to the treatment of the constant sampling rate in [28]. This way both stabilization and  $H^2$  optimization problems can be solved, thus justifying the predictor-based structure proposed in [13] without a proof. This approach will not work in the  $H^\infty$  case though. Another alternation that seems to be immediate is to apply the ideas of this paper to the formulation proposed in [9, Ch. 4], where the analog loop is closed not only instantaneously, but rather during some short time intervals. A more laborious extension would be to come up with a theoretically justified event generation mechanism.

## APPENDIX

### A. Proof of Theorem 5.1

We start with the following technical result:

*Lemma A.1:* Let  $J_0$  be given by (4'') with  $F$  and  $L$  as in the statement of Theorem (5.1). Consider the family of controllers  $\mathcal{F}_1(J_0, Q)$  for a causal linear  $Q$  such that  $\|Q\|_2 < \infty$ . Then

$$\|T_{zw}\|_2^2 = \gamma_0^2 + \|Q\|_2^2,$$

where  $\gamma_0$  is the optimal  $H^2$  performance attainable by continuous-time controllers.

*Proof:* The closed-loop map for the considered family of controllers is [16, Thm. 12.16]  $T_{zw} = T_1 + T_2 Q T_3$ , where

$$\begin{bmatrix} T_1(s) & T_2(s) \\ T_3(s) & 0 \end{bmatrix} = \begin{bmatrix} A_F & -B_u F & B_w & B_u \\ 0 & A_L & B_L & 0 \\ C_F & -D_{zu} F & 0 & D_{zu} \\ 0 & C_y & D_{yw} & 0 \end{bmatrix}$$

with Hurwitz  $A_F := A + B_u F$  and  $A_L := A + L C_y$ ,  $B_L := B_w + L D_{yw}$ , and  $C_F := C_z + D_{zu} F$ . Moreover,  $T_1 \in H^2$ ,  $T_2$  is inner [16, Thm. 13.32] and  $T_3$  is co-inner [16, Thm. 13.35].

Now, (15) defines a (degenerate) Hilbert space with the inner product

$$\langle G_1, G_2 \rangle_2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\tau^\infty \text{tr}(g_2'(t, \tau) g_1(t, \tau)) dt d\tau,$$

so that  $\|G\|_2^2 = \langle G, G \rangle_2$ . If  $G$  is a causal LTI system, its adjoint with respect to the inner product above,  $G^*$ , is the anti-causal LTI system, whose transfer function equals  $[G(-s)]'$ , exactly as in the case of the conventional  $H^2$  space. We then have:

$$\begin{aligned} \|T_{zw}\|_2^2 &= \langle T_1 + T_2 Q T_3, T_1 + T_2 Q T_3 \rangle_2 \\ &= \|T_1\|_2^2 + \|T_2 Q T_3\|_2^2 + 2 \text{Re} \langle T_2 Q T_3, T_1 \rangle_2 \\ &= \|T_1\|_2^2 + \|Q\|_2^2 + 2 \text{Re} \langle Q, V \rangle_2, \end{aligned}$$

where  $V := T_2^* T_1 T_3^*$  and the facts that  $T_2^* T_2 = I$  and  $T_3 T_3^* = I$  were used. It can be verified, via straightforward state-space manipulations, that  $V$  is anti-causal, with

$$V(s) = \begin{bmatrix} -A_F' & -A_F' X Y - X A Y & X L \\ 0 & -A_L' & C_y' \\ B_u' & -D_{zu}' C_z Y & 0 \end{bmatrix},$$

where  $X \geq 0$  and  $Y \geq 0$  are the stabilizing solutions of the state-feedback and filtering Riccati equations, respectively. This implies that the responses of  $V$  and  $Q$  to the same impulse have disjoint supports. Therefore,  $\langle Q, V \rangle_2 = 0$ , which completes the proof (with  $\gamma_0 = \|T_1\|_2$ ).  $\square$

By Lemma 4.1, the controller of the form  $\mathcal{F}_1(J_0, Q)$  is a sampled-data one iff  $Q = Q_{\text{stat}} + Q_{\text{sd}}$  for a given  $Q_{\text{stat}}$  and any stable sampled-data  $Q_{\text{sd}}$ . Remember that the lifting of  $Q_{\text{stat}}$  is static and the lifting of  $Q_{\text{sd}}$  is strictly proper. Therefore, the impulse responses of  $Q_{\text{stat}}$  and  $Q_{\text{sd}}$  are non-overlapping for any admissible  $Q_{\text{sd}}$ , which, in turn, implies that

$$\|Q\|_2^2 = \|Q_{\text{stat}} + Q_{\text{sd}}\|_2^2 = \|Q_{\text{stat}}\|_2^2 + \|Q_{\text{sd}}\|_2^2.$$

Thus, the optimal performance is attained with any  $Q$  such that  $Q - Q_{\text{stat}}$  is in the kernel of semi-norm (15).

Compute now  $\|Q_{\text{stat}}\|_2^2$ . By (8),  $Q_{\text{stat}}$  can be described by

$$\begin{aligned} \dot{x}_Q(t) &= Ax_Q(t) + L\epsilon(t), & x_Q(t_i) &= 0 \\ \eta_Q(t) &= Fx_Q(t) \end{aligned} \quad (20)$$

Its impulse response is  $q_{\text{stat}}(t, \tau) = Fe^{A(t-\tau)}L\mathcal{K}_{[\tau, t_j]}(t)$ , where  $t_j$  is the smallest element of  $\{t_i\}$  such that  $t_j \geq \tau$  and  $\mathcal{K}_{[a, b]}(t)$  is the characteristic function of the interval  $[a, b)$ . Then

$$\begin{aligned} \|Q_{\text{stat}}\|_2^2 &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} \int_{\tau}^{\infty} \|q_{\text{stat}}(t, \tau)\|_F^2 dt d\tau \\ &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\tau}^{\infty} \|q_{\text{stat}}(t, \tau)\|_F^2 dt d\tau \\ &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\tau}^{t_{j+1}} \|Fe^{A(t-\tau)}L\|_F^2 dt d\tau, \end{aligned}$$

from which the expression for the achievable performance follows by straightforward integration variable change.

Finally, the optimal control law is in form (13) because  $K_0$  is observer based. ■

### B. Proof of Theorem 5.3

In addition to the notation introduced prior to the formulation of the Theorem, define

$$\tilde{B}_u := B_u + \gamma^{-2}YC_z' D_{zu}, \quad \tilde{C}_y := C_y + \gamma^{-2}D_{yw}B_w' X,$$

and  $Z_\gamma := (I - \gamma^{-2}YX)^{-1}$ . It is known [16, Thm. 16.5] that if  $\gamma > \gamma_{\text{opt}}$ , all  $\gamma$ -suboptimal LTI controllers can be characterized as  $\mathcal{F}_1(J_\gamma, Q)$  for

$$J_\gamma(s) = \left[ \begin{array}{c|cc} A_\gamma & -Z_\gamma L & Z_\gamma \tilde{B}_u \\ \hline F & 0 & I \\ \hline -\tilde{C}_y & I & 0 \end{array} \right] \quad (21)$$

and an arbitrary LTI  $Q \in H^\infty$  such that  $\|Q\|_\infty < \gamma$ , where  $A_\gamma := A + \gamma^{-2}B_w B_w' X + B_u F + Z_\gamma L \tilde{C}_y$ . Because the central controller is the one corresponding to  $Q = 0$ ,  $K_0 = \mathcal{F}_1(J_\gamma, 0)$ . The parametrization above extends to time-varying controllers as well. Namely, the set of all  $\gamma$ -suboptimal linear causal controllers is  $\mathcal{F}_1(J_\gamma, Q)$ , where  $Q$  is an arbitrary bounded causal operator on  $L^2(\mathbb{R}^+)$  such that its induced norm  $\|Q\| < \gamma$ , see the arguments in [29].

By Lemma 4.1, a controller of the form  $\mathcal{F}_1(J_\gamma, Q)$  is in the sampled-data form iff  $Q = Q_{\text{stat}} + Q_{\text{sd}}$  for a  $Q_{\text{stat}}$ , verifying

$$\begin{aligned} \dot{x}_Q(t) &= A_\gamma^\times x_Q(t) + Z_\gamma L \epsilon(t), & x_Q(t_i) &= 0 \\ \eta(t) &= Fx_Q(t) \end{aligned} \quad (22)$$

where  $A_\gamma^\times := A_\gamma - Z_\gamma(\tilde{B}_u F + L \tilde{C}_y) = A + \gamma^{-2}(B_w B_w' X + Z_\gamma Y F' F)$ , and any stable causal sampled-data  $Q_{\text{sd}}$ . The existence of an admissible  $Q$  is then equivalent to the existence of a causal sampled-data system  $Q_{\text{sd}}$  such that  $\|Q_{\text{stat}} + Q_{\text{sd}}\| < \gamma$ . To address the latter, the following result is required:

**Lemma A.2:**  $\|Q_{\text{stat}} + Q_{\text{sd}}\| \geq \|Q_{\text{stat}}\|$  for all causal sampled-data systems  $Q_{\text{sd}}$ .

*Proof:* In the lifted domain,  $\check{Q}_{\text{stat}}$  is static and  $\check{Q}_{\text{sd}}$  is strictly causal. Hence, the responses of  $\check{Q}_{\text{stat}}$  and  $\check{Q}_{\text{sd}}$  to any input  $\check{\epsilon}$  such that  $\check{\epsilon}[i] = 0$  for all  $i \neq j$  for some given  $j \in \mathbb{Z}^+$  are non-overlapping (zeros  $\forall i \neq j$  and  $\forall i \leq j$ , respectively). As a result, in the time domain we have that for any  $\epsilon(t)$  with support in  $[t_i, t_{i+1})$ ,

$$\|(Q_{\text{stat}} + Q_{\text{sd}})\epsilon\|_2^2 = \|Q_{\text{stat}}\epsilon\|_2^2 + \|Q_{\text{sd}}\epsilon\|_2^2 \geq \|Q_{\text{stat}}\epsilon\|_2^2.$$

where  $\|\cdot\|_2$  stands for the  $L^2(\mathbb{R}^+)$  signal norm. The result then follows by observing that the worst-case input for  $Q_{\text{stat}}$  has support in  $[t_i, t_{i+1})$  for some  $i$ , which, in turn, is a consequence of the fact that  $Q_{\text{stat}}$  resets at each  $t_i$  (by Lemma 2.2). □

It follows from Lemma A.2 that an admissible  $Q$  exists iff  $\|Q_{\text{stat}}\| < \gamma$  (as we can always pick  $Q_{\text{sd}} = 0$ ). The norm bound can then be verified by the following result:

**Lemma A.3:** Let  $\gamma > \gamma_{\text{opt}}$  and  $Q_{\text{stat}}$  be given by (22). Then  $\|Q_{\text{stat}}\| < \gamma$  iff the conditions of the Theorem hold.

*Proof:* It is readily seen that  $\|Q_{\text{stat}}\| < \gamma$  iff the  $L^2[0, h_i]$ -induced norm of  $\mathcal{F}_u(J_\gamma^{-1}(s), 0) = F(sI - A_\gamma^\times)^{-1}Z_\gamma L$  is less than  $\gamma$  for all  $i \in \mathbb{Z}^+$ . But the  $L^2[0, h]$ -induced norm of an LTI system is a monotonically increasing function of  $h$ . Hence, we only need to check the norm for the maximal  $h_i$ .

It is known [30, Lem. 2.2] that the  $L^2[0, h]$ -induced norm of  $\mathcal{F}_u(J_\gamma^{-1}, 0)$  is less than  $\gamma$  iff the differential Riccati equation

$$\dot{R}(t) = A_\gamma^\times R(t) + R(t)(A_\gamma^\times)' + Z_\gamma L L' Z_\gamma' + \gamma^{-2}R(t)F'FR(t)$$

with  $R(0) = 0$  has a bounded solution in the whole interval  $[0, h]$ . This Riccati equation, in turn, is associated with the Hamiltonian matrix [30, Lem. 2.3]

$$H_R := \begin{bmatrix} -(A_\gamma^\times)' & -\gamma^{-2}F'F \\ Z_\gamma L L' Z_\gamma' & A_\gamma^\times \end{bmatrix}.$$

It can be shown [31, Eqn. (14)] that

$$H_R = \begin{bmatrix} Z_\gamma' & \gamma^{-2}X \\ YZ_\gamma' & I \end{bmatrix}^{-1} H_P \begin{bmatrix} Z_\gamma' & \gamma^{-2}X \\ YZ_\gamma' & I \end{bmatrix},$$

where

$$H_P := \begin{bmatrix} -A' & -\gamma^{-2}C_z' C_z \\ B_w B_w' & A \end{bmatrix}$$

is the Hamiltonian matrix associated with  $P(t)$ . As a result,

$$R(t) = (I - \gamma^{-2}P(t)X)^{-1}(P(t) - Y)Z_\gamma',$$

so that it is bounded iff  $\det(I - \gamma^{-2}P(t)X) \neq 0$ . It is readily seen that  $P_d(t) := \dot{P}(t)$  satisfies the Lyapunov equation

$$\dot{P}_d(t) = A_P(t)P_d(t) + P_d(t)A_P'(t), \quad P_d(0) = LL' \geq 0$$

where  $A_P := A + \gamma^{-2}PC_z' C_z$ . Hence,  $\dot{P}(t) \geq 0$  for all  $t$  and  $P(t)$  is non-decreasing. We also know that  $\rho(P(0)X) < \gamma^2$  whenever  $\gamma > \gamma_{\text{opt}}$ . Thus, the boundedness of  $R(t)$  in  $[0, h]$  is equivalent to  $\rho(P(t)X) < \gamma^2$  at each  $t$  in this interval. □



To complete the proof of the Theorem, we only need to show that controller (17) is a particular case of (7) if  $J_0 = J_\gamma$ . This can be verified by direct substitution using the fact that

$$A_\gamma - Z_\gamma \tilde{B}_u F = Z_\gamma A_L Z_\gamma^{-1},$$

which can be verified via some lengthy algebra. ■

#### ACKNOWLEDGMENTS

I am indebted to Igor Gindin and Miriam Zacksenhouse for drawing my attention to the problem via [13].

#### REFERENCES

- [1] K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1997.
- [2] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. London: Springer-Verlag, 1995.
- [3] P. T. Kabamba, "Control of linear systems using generalized sampled-data hold functions," *IEEE Trans. Automat. Control*, vol. 32, no. 9, pp. 772–783, 1987.
- [4] R. A. Yackel, B. C. Kuo, and G. Singh, "Digital redesign of continuous systems by matching of states at multiple sampling periods," *Automatica*, vol. 10, no. 1, pp. 105–111, 1974.
- [5] A. Feuer and G. C. Goodwin, "Generalized sample hold function: Frequency domain analysis of robustness, sensitivity and intersample difficulties," *IEEE Trans. Automat. Control*, vol. 39, no. 5, pp. 1042–1045, 1994.
- [6] G. Tadmor, " $H^\infty$  optimal sampled-data control in continuous-time systems," *Int. J. Control*, vol. 56, no. 1, pp. 99–141, 1992.
- [7] L. Mirkin, H. Rotstein, and Z. J. Palmor, " $H^2$  and  $H^\infty$  design of sampled-data systems using lifting. Part I: General framework and solutions," *SIAM J. Control Optim.*, vol. 38, no. 1, pp. 175–196, 1999.
- [8] K. J. Åström, "Event based control," in *Analysis and Design of Nonlinear Control Systems*, A. Astolfi and L. Marconi, Eds. Berlin: Springer-Verlag, 2008, pp. 127–147.
- [9] E. Garcia, P. J. Antsaklis, and L. A. Montestruque, *Model-Based Control of Networked Systems*. Boston: Birkhäuser, 2014.
- [10] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proc. 51st IEEE Conf. Decision and Control*, Maui, HI, 2012, pp. 3270–3285.
- [11] P. J. Gawthrop and L. Wang, "Intermittent redesign of continuous controllers," *Int. J. Control*, vol. 83, pp. 1581–1594, 2010.
- [12] B. Friedland and W. Grossman, "On controlling continuous-time processes with data on occurrence of discrete events," in *Proc. 4th IEEE Conf. Control Applications*, Albany, NY, 1995, pp. 736–741.
- [13] P. Gawthrop, I. Loram, M. Lakie, and H. Gollee, "Intermittent control: a computational theory of human control," *Biol. Cybern.*, vol. 104, pp. 31–51, 2011.
- [14] L. Mirkin and H. Rotstein, "On the characterization of sampled-data controllers in the lifted domain," *Syst. Control Lett.*, vol. 29, no. 5, pp. 269–277, 1997.
- [15] D. McFarlane and K. Glover, *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*, ser. Lecture Notes in Control and Inform. Sci. Berlin: Springer-Verlag, 1990, vol. 138.
- [16] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
- [17] A. Feintuch and R. Sacks, *System Theory: A Hilbert Space Approach*. New York: Academic Press, 1982.
- [18] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Trans. Automat. Control*, vol. 26, no. 2, pp. 301–320, 1981.
- [19] S. Boyd, C. Barratt, and S. Norman, "Linear controller design: Limits of performance via convex optimization," *Proc. IEEE*, vol. 78, no. 3, pp. 529–574, 1990.
- [20] J. C. Willems, *The Analysis of Feedback Systems*. Cambridge, MA: The MIT Press, 1971.
- [21] D. Lehmann and J. Lunze, "Event-based output-feedback control," in *Proc. 19th IEEE Mediterranean Conf. on Control and Automation*, Corfu, Greece, 2011, pp. 982–987.
- [22] L. A. Montestruque and P. J. Antsaklis, "Stability of model-based networked control systems with time-varying transmission times," *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1562–1572, 2004.
- [23] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.
- [24] V. Ionescu and M. Weiss, "The  $l^2$ -control problem for time-varying discrete systems," *Syst. Control Lett.*, vol. 18, pp. 371–381, 1992.
- [25] A. Ichikawa and H. Katayama, *Linear Time Varying Systems and Sampled-data Systems*, ser. Lecture Notes in Control and Inform. Sci. Berlin: Springer-Verlag, 2001, vol. 265.
- [26] E. Bini and G. M. Buttazzo, "The optimal sampling pattern for linear control systems," *IEEE Trans. Automat. Control*, vol. 59, no. 1, pp. 78–90, 2014.
- [27] K. J. Åström and B. Bernhardsson, "Systems with Lebesgue sampling," in *Directions in Mathematical Systems Theory and Optimization*, ser. Lecture Notes in Control and Inform. Sci., A. Rantzer and C. I. Byrnes, Eds. London: Springer-Verlag, 2003, vol. 286, pp. 1–13.
- [28] L. Mirkin, T. Shima, and G. Tadmor, "Sampled-data  $H^2$  optimization of systems with I/O delays via analog loop shifting," *IEEE Trans. Automat. Control*, vol. 59, no. 3, pp. 787–791, 2014.
- [29] G. Tadmor, "Worst case design in the time domain: The maximum principle and the standard  $H^\infty$  problem," *Math. Control, Signals and Systems*, vol. 3, pp. 301–324, 1990.
- [30] G. Gu, J. Chen, and O. Toker, "Computation of  $\mathcal{L}_2[0, h]$  induced norms," in *Proc. 35th IEEE Conf. Decision and Control*, Kobe, Japan, 1996, pp. 4046–4051.
- [31] L. Mirkin, "On the extraction of dead-time controllers and estimators from delay-free parametrizations," *IEEE Trans. Automat. Control*, vol. 48, no. 4, pp. 543–553, 2003.